A graph with quantum symmetry and finite quantum automorphism group

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joint work with David Roberson

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Colored graphs

- Consider a finite graph $G = (V, E)$ without multiple edges, i.e. V finite set and $E \subseteq V \times V$
- A *colored* graph is a graph G along with a coloring function $c : V \cup E \to S$ for some set S.

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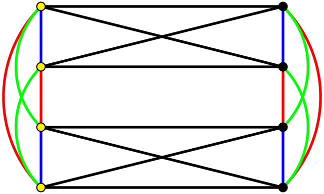

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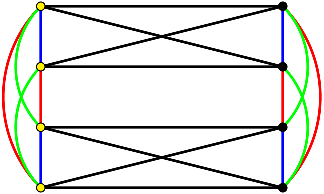

- Adjacency matrices $A_{G_c} \in M_n(\{0,1\})$, where $(A_{G_c})_{ij} = \begin{cases} 1 \text{ if } (i,j) \in E_c \\ 0 \text{ otherwise} \end{cases}$
- A graph automorphism is a bijection $\sigma : V \to V$ such that $(i,j) \in E_c$ if and only if $(\sigma(i), \sigma(j)) \in E_c$, where additionally $c(i) = c(\sigma(i))$.
- Automorphism group $\mathrm{Aut}(G) = \{\sigma \in S_n \mid \sigma A_{G_c} = A_{G_c}\sigma \text{ and } \sigma_{ij} = 0 \text{ if } c(i) \neq c(j)\}$

The quantum symmetric group

Definition (Wang, 1998)

The *quantum symmetric group* $S_n^+ = (C(S_n^+), u)$ is the compact matrix quantum group, where

$$C(S_n^+) := C^*(u_{ij},\, 1 \leq i,j \leq n \,|\, u_{ij} = u_{ij}^* = u_{ij}^2,\, \sum_k u_{ik} = \sum_k u_{ki} = 1).$$

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- The C^*-algebra $C(S_n^+)$ is commutative for $n \leq 3$ and non-commutative for $n \geq 4$
- For $n = 4$, the C^*-algebra $C(S_4^+)$ is non-commutative because of the surjective *-homomorphism

$$\varphi : C(S_4^+) \to C^*(p, q, 1 \,|\, p = p^* = p^2, q = q^* = q^2),$$
$$u \mapsto \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}.$$

Quantum automorphism groups of colored graphs

Definition

Let $G = (V, E)$ be a colored graph. The *quantum automorphism group* $Qut(G)$ is the compact matrix quantum group $(C(Qut(G)), u)$, where $C(Qut(G))$ is the universal C^*-algebra with generators u_{ij} fulfilling

$$\begin{aligned}
&u_{ij} = u_{ij}^* = u_{ij}^2, && i, j \in V(G)\\
&\sum_k u_{ik} = \sum_k u_{ki} = 1, && i \in V(G)\\
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&uA_{G_c} = A_{G_c}u && \text{for all edge colors } c.
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Here $uA_{G_c} = A_{G_c}u$ is nothing but $\sum_k u_{ik}(A_{G_c})_{kj} = \sum_k (A_{G_c})_{ik}u_{kj}$.

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- For uncolored graphs, quantum automorphism groups were defined by Banica in 2005.
- The graph G has *no quantum symmetry* if $C(Qut(G))$ is commutative, or equivalently $C(Qut(G)) = C(\mathrm{Aut}(G))$. Otherwise, the graph G *does have quantum symmetry*.

Linear constraint systems and their solution groups

Definition

Let $M \in \mathbb{F}_2^{m \times n}$ and $b \in \mathbb{F}_2^m$ with $b \neq 0$. The *solution group* $\Gamma(M, b)$ of the linear system $Mx = b$ is the group generated by elements x_i for $i \in [n]$ and an element J satisfying the following relations:

(1) $x_i^2 = 1$ for all $i \in [n]$;

(2) $x_i x_j = x_j x_i$ if there exists $k \in [m]$ s.t. $M_{ki} = M_{kj} = 1$;

(3) $\prod_{i:M_{ki}=1} x_i = J^{b_k}$ for all $k \in [m]$;

(4) $J^2 = 1$;

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$$C^*(\Gamma) = C^*\left(x_i \,|\, x_i = x_i^*, x_i^2 = 1, \prod_{i: M_{ki}=1} x_i = 1, x_i x_j = x_j x_i \text{ if } M_{ki} = M_{kj} = 1 \text{ for some } k\right)$$

LCS from connected graphs

Let H be a connected graph with vertex set $[m]$ and label the edges $1, \dots, n := |E(H)|$.
Let $M_H \in \mathbb{F}_2^{m \times n}$ be the matrix, where

$$(M_H)_{ki} = \begin{cases} 1 & \text{if } k \in V(H) \text{ is incident to } i \in E(H); \\ 0 & \text{o.w.} \end{cases}$$

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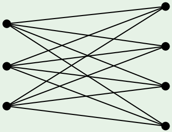
Example

Graph $K_{3,4}$



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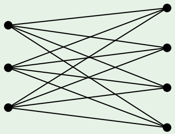

Linear constraint system

$$x_1 + x_2 + x_3 + x_4 = 0, \qquad x_1 + x_5 + x_9 = 0,$$
$$x_5 + x_6 + x_7 + x_8 = 0, \qquad x_2 + x_6 + x_{10} = 0,$$
$$x_9 + x_{10} + x_{11} + x_{12} = 0, \qquad x_3 + x_7 + x_{11} = 0,$$
$$x_4 + x_8 + x_{12} = 0.$$

A colored graph associated to the LCS

Let H be a connected graph, $M_H \in \mathbb{F}_2^{m \times n}$ as before. Define the colored graph $G := G(M_H, 0)$ as follows. Let $S_k = \{i \in [n]; (M_H)_{ki} = 1\}$.

(1) Vertices: $\left\{(k, \alpha) : k \in [m], \alpha : S_k \to \{\pm 1\}, \prod_{i \in S_k} \alpha_i = 1\right\}$. The color of a vertex $v = (k, \alpha)$ is k.

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- the graph has $3 \times 8 + 4 \times 4 = 40$ vertices and seven vertex-colors,

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Example

For $H = K_{3,4}$:

- the graph has $3 \times 8 + 4 \times 4 = 40$ vertices and seven vertex-colors,
- edges between vertices associated to the same equation may have different colors,
- edges between vertices associated to different equations always have color -1.

Quantum automorphism group of the colored graph

Theorem (Roberson, S. 2021)

Let $M \in \mathbb{F}_2^{m\times n}$. Set $G = G(M,0)$ and $\Gamma = \Gamma(M,0)$. Then there exists a $*$-isomorphism $\varphi : C^*(\Gamma) \to C(Qut(G))$ such that $\Delta_G \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_\Gamma$.

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- By coloring the graph in this specific way, we make sure that those are the only quantum automorphisms of the graph

Decoloring the vertices of $G(M, 0)$

Let G be a vertex – and edge-colored graph.

(1) Attach a path of length $n_c \in \mathbb{N}_0$ to every vertex colored c, where $n_{c_1} \neq n_{c_2}$ for colors $c_1 \neq c_2$ and then decolor the vertices of the graph.

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We denote this new edge-colored (but not vertex-colored) graph by G'.

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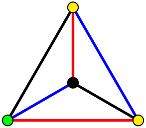
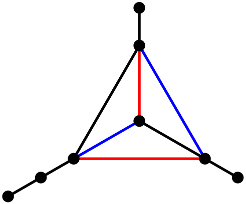
$G =$ 

$G' =$ 

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Let H be a connected graph with $\deg(v) \geq 2$ for all $v \in V(H)$. Let $G := G(M_H, 0)$ as before and construct G'. Then there exists a $*$-isomorphism $\varphi : C(Qut(G)) \to C(Qut(G'))$ such that $\Delta_{G'} \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_G$.

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- Use the following result: If $\deg(v) \neq \deg(w)$, then $u_{vw} = 0$.

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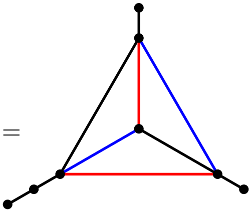
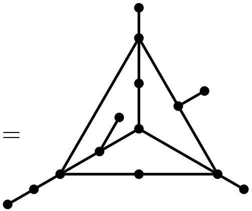
$G' =$ 

$G'' =$ 

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Let H be a connected graph with $\deg(v) \geq 2$ for all $v \in V(H)$. Let $G := G(M_H, 0)$ as before and construct G'' from G', where we choose $c_0 = -1$. Then there exists a $*$-isomorphism $\varphi : C(Qut(G')) \to C(Qut(G''))$ such that $\Delta_{G''} \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_{G'}$.

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Denote by e_i the vertices in the added path to the subdivision e_0 of e, with $d(e, e_i) = i$

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- Show:
  (1) $u_{e_i f_j} = 0$ for $i \neq j$ and $u_{e_i v} = 0$
  (2) $u_{e_i f_i} = u_{vx} u_{wy} + u_{vy} u_{wx}$ for $e = (v, x)$ and $f = (w, y)$,
  (3) $u_{e_i f_i} = 0$ for $c(e) \neq c(f)$.

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- Also need to show $u'_{vx} u'_{wy} = u'_{wy} u'_{vx}$ for $(v, w), (x, y) \in E(G')$, $c(v, w) \neq c_0 \neq c(x, y)$, where u' fundamental representation of $Qut(G')$

A graph with quantum symmetry and finite quantum automorphism group

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David E. Roberson and Simon Schmidt. "Solution group representations as quantum symmetries of graphs" arXiv:2111.12362 (2021)